Approximate Calculation of an Ephemeris in Unperturbed Elliptic Motion

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Introductory Remarks

THE problem concerning unperturbed motion, i.e., the two-body problem, has an exact solution that is actually used also in the computation of an ephemeris without the calculation of perturbations. This solution is not considered simple enough because there is no direct connection of the coordinates with time. Therefore, it is worth while to attempt to simplify the calculation of ephemerides.

The relative complexity of the solution of the differential equations of the two-body problem depends on the nonlinearity of the equations. In order to simplify the solution, it is necessary to eliminate the nonlinearity. From the time of Gauss, the method of averaging has been used for the simplification of the equations of motion in celestial mechanics. The averaging was subject to the force function of the problem. It is shown that it gives the possibility of finding the secular and long period perturbations. It should be noted that the question concerning averaging is not so clear as it sometimes seems. It is sufficient to cite a very simple reason: If, in the two-body problem, the average value of the force function is obtained with respect to the mean anomaly, then we shall obtain a constant value, and the approximate differential equations with the average force function determine straight line motion with constant velocity. In the present article, the linearization of the differential equations of the two-body problem is considered, being based on a partial average of the differential equations of motion. The linearity of the equations is "violated" by the presence of the factor $1/r^3$ in all of the equations; thus the obvious method of linearization will be the replacement of this factor by its average value if the motion is considered along an ellipse.

1 The Mean Distance and Mean Function of Distance

The concept of mean distance presupposes the existence of a set of distances from which the mean value is formed as one from the summary of characteristics of all of the set. Under the mean we shall imply the mathematical expectation of the distance or its function found according to the rules of the theory of probability. The concept of the mean distance is to a certain extent conventional, because it depends both on the choice of the basic random quantity and on the assumptions about the probability density of this basic quantity. The semimajor axis of the elliptic orbit is the element usually called the mean distance.

We shall consider certain methods for the determination of the mean distance.

a) We shall assume the distance itself for the basic random quantity, and we shall make an assumption concerning the law of uniform distribution. Since, in elliptic motion, the distance takes on values from $a(1 - e)$ to $a(1 + e)$, the probability density equals $1/(2\,ae)$, and the mean value will be a.

b) Motion along an elliptic orbit occurs nonuniformly, not agreeing with the assumption about the uniform law of distribution. If the ephemeris is to represent the calculation of the whole period of rotation of the planet around the sun, then large distances will be met more often than small distances. Thus it is possible to assume arbitrarily such a probability density of distances which will grow monotonically for increases of r. We shall make the simplest assumption concerning the linear law of probability density. Considering the difference $f[a(1 + e)] - f[a(1 - e)] = \gamma$ a parameter of the problem, we shall obtain the probability density in the form $f(r) = 1/(2\,ae) + \gamma(r - a)/2\,ae$. We consider the parameter γ to be a function of the eccentricity of the orbit, which it is possible to select so as to obtain the natural value of the mean in the extreme cases ($e = 0$ or $e = 1$). The mean value of the distance is determined according to the formula

$$\bar{r} = a + \gamma e^2a^2/3$$

Here it is seen that for $e = 0$ the natural result $\bar{r} = a$ is obtained. According to the meaning of the problem, γ must have the dimensions of $1/a$ in order to secure the dimension of probability density. Taking this into account where $\gamma = 1/(ae)$ we shall obtain $\bar{r} = a + ea/3$.

This result has a methodical value, because it shows that for the simplest assumptions concerning the probability density of r, the mean value of the heliocentric distance is not equal to a.

It is possible to take a basic random quantity on which distance depends, to give for this quantity a probability density, and to determine the mean distance as the mathematical expectation function of the basic random quantity. The distance can be considered a function of one of the three commonly used anomalies, each of which can be taken for the auxiliary random quantity. Correspondingly, we shall obtain three more versions of the mean value of the heliocentric distance. For each of the versions, it is easy to find, according to known rules, the mean value of the function of distance.

c) Let us assume that all correct directions to the planet are equiprobable, that is, the true anomaly is uniformly distributed. We shall determine the mean distance of the function of the true anomaly for which the given probability density equals $1/(2\pi)$. According to known rules, we obtain

$$\bar{r} = \frac{p}{2\pi}\int_{-\pi}^{\pi}\frac{dv}{1 + e\cos v} = \frac{p}{\pi}\int_{0}^{\pi}\frac{dv}{1 + e\cos v} = a\sqrt{1 - e^2} = b$$

Here the mean distance determined is less than a.

d) We take for the basic random quantity the eccentric anomaly and assume equal probability of the values of E from 0 up to 2π, that is, we consider the probability density equal to $1/(2\pi)$. By making use of the known expression of distance by the eccentric anomaly, we obtain

$$\bar{r} = \frac{a}{2\pi}\int_{0}^{2\pi}(1 - e\cos E)dE \qquad \bar{r} = a$$

Noting again the mean value of the factor $1/r^3$ entering in all

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differential equations and "violating" the nonlinearity, we obtain

$$\overline{\left(\frac{1}{r^3}\right)} = \frac{1}{\pi a^3}\int_0^{\pi} \frac{dE}{(1 - e\cos E)^3} = \frac{1}{a^3}\cdot\frac{1+\frac{1}{2}e^2}{(1-e^2)^{5/2}} = (1 + 3e^2 + \ldots)\frac{1}{a^3}$$

e) It is most natural to consider time as the basic random quantity, distributed uniformly during the period of rotation of the planet around the sun. It is much more convenient to take the mean anomaly instead of time for the basic quantity, because it takes values from 0 up to 2π for all planets. Consequently, the probability density of quantity M always equals $1/(2\pi)$. The distance and its function are expressed by time, that is, through the mean anomaly by an infinite series. The coefficients of this series are functions of the eccentricity, also being represented by infinite series. Thus, for example (1)[2]

$$\frac{r}{a} = 1 + \frac{1}{2}e^2 - e\sum_{k=1}^{\infty}\frac{2}{k}I_k'(ke)\cos kM$$

$$\frac{a}{r} = 1 + 2\sum_{k=1}^{\infty} I_k(ke)\cos kM$$

where I and I' are Bessel functions and their derivatives. The method of averaging, using such series, gives simpler terms for the series not containing the mean anomaly. We obtain

$$\bar{r} = a\left(1 + \frac{e^2}{2}\right) \qquad \overline{\left(\frac{1}{r}\right)} = \frac{1}{a}$$

Comparison with item b shows that the linear law of probability density with the parameter $\gamma = 3/2a$ may be taken for the distance, if only the mean distance is required. The mean value of the square of the heliocentric distance is easily determined by making use of the series (1):

$$\left(\frac{r}{a}\right)^2 = 1 + \frac{3}{2}e^2 - 4\sum_{k=1}^{\infty}\frac{I_k(ke)}{k^2}\cos kM$$

from which we obtain: $\overline{r^2} = a^2(1 + 3/2\cdot e^2)$.

It is possible also to construct analogous infinite series for other powers of the distance, but they are inconvenient for the determination of mean values, because the collections of terms in them which are not dependent on time also represent infinite series.

We shall apply other methods. The mean anomaly is very simply expressed through the eccentric anomaly by Kepler's equation. This makes it possible to determine easily the probability density of the eccentric anomaly by the probability density of the mean anomaly:

$$f_1(E) = \frac{1 - e\cos E}{2\pi} = \frac{r}{2\pi a}$$

Analogously it is possible to determine the probability density of the distance by the method

$$f_2(r) = r/[2\pi a\sqrt{a^2e^2 - (a - r)^2}]$$

A U-shaped distribution is obtained. The geometric mean of the deviations from the perihelion and aphelion distances is in the denominator. From here the value of the mean distance found previously is determined easily according to the formula

$$\bar{r} = 2\int_{a(1-e)}^{a(1+e)} rf_2(r)dr = a\left(1 + \frac{1}{2}e^2\right)$$

The factor 2 must be introduced, because the limits of integration correspond to only half of the trajectory. It is still simpler to consider r a function of E, and for the determination of the mathematical expectation $\phi(r)$ to calculate the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\phi[1(1 - e\cos E)](1 - e\cos E)dE$$

It is possible to verify all previous means by such a method and to obtain new expressions:

$$\overline{r^3} = a^3(1 + 3e^2 + \tfrac{3}{8}e^4)\ \ldots\ \text{(exactly)}$$

$$\overline{\left(\frac{1}{r^2}\right)} = \frac{1}{a^2\sqrt{1 - e^2}} = \frac{1}{ab}$$

$$\overline{\left(\frac{1}{r^3}\right)} = \frac{1}{a^3(1 - e^2)^{3/2}} = \frac{1}{b^3}$$

2 The Simplest Approximation Formula

The semimajor axis of an elliptic orbit also can be considered the mean distance for certain assumptions (uniform distribution of distances or eccentric anomalies), although, for these assumptions, a unit divided by the cube of the distance does not have the same mean value as a unit divided by the cube of the semimajor axis. However, for maximum simplicity of the approximation, r^{-3} may be replaced by a^{-3}. The system of differential equations is reduced to three independent linear differential equations of the second order which are easily solved according to the initial coordinates and their derivatives. Let x_0, y_0, z_0 be the initial values of the coordinates at the moment that we assume equal to zero, and let $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$, be the components of initial velocity. In the differential equations we replace the factor k^2/a^3 by the square of the mean motion (the units are the usual ones in the astronomical problem) μ^2, and we obtain the solution of all the equations in the form

$$\bar{x} = x_0\cos\mu t + \frac{\dot{x}_0}{\mu}\sin\mu t$$

$$\bar{y} = y_0\cos\mu t + \frac{\dot{y}_0}{\mu}\sin\mu t$$

$$\bar{z} = z_0\cos\mu t + \frac{\dot{z}_0}{\mu}\sin\mu t$$

If the direction cosines are denoted by P_x, P_y, P_z, Q_x, Q_y, Q_z, then, by the given elements, referred to epoch t_0, we find the coefficients of the solution according to known formulas (2). The orbital coordinates are

$$\xi_0 = a(\cos E_0 - e) \qquad \eta_0 = b\sin E_0$$

$$b = a\cos\phi \qquad x_0 = P_x\xi_0 + Q_x\eta_0$$

$$C_x = \left(\frac{\dot{x}_0}{\mu}\right) = \left(\frac{a}{r_0}\right)(-aP_x\sin E_0 + bQ_x\cos E_0)$$

The quantities y_0, z_0, C_y, C_z are determined analogously. In order to put together a representation from the trajectory, we eliminate time from the equations of motion and obtain the equation

$$\begin{vmatrix} \bar{x} & \bar{y} & \bar{z} \\ x_0 & y_0 & z_0 \\ C_x & C_y & C_z \end{vmatrix} = 0$$

From this equality it is clear that the motion is planar. Since the initial position and initial velocity are exact, and the approximated trajectory is general, then the approximated orbit is situated on one plane with the exact orbit and is tangent to it at the initial point. For the determination of the form of

[2] Numbers in parentheses indicate References at end of paper.

the approximated orbit, we assume that the fundamental plane of the coordinates $\bar{x}\bar{y}$ coincides with the plane of the orbit. Eliminating time, we obtain the equation of the orbit:

$$(C_y^2 + y_0^2)\bar{x}^2 - 2(C_xC_y + x_0y_0)\bar{x}\bar{y} + (C_x^2 + x_0^2)\bar{y}^2 = (C_yx_0 - C_xy_0)^2$$

from which it is seen that the approximated orbit is an ellipse with the center at the origin of the coordinates, that is, at the central point. This approximated orbit differs essentially from the exact ellipse, the focus of which is located at point of the center of mass. For the calculation of the semi-axis of the ellipse, we have the equation

$$S^2 - (x_0^2 + y_0^2 + C_x^2 + C_y^2)S + (C_yx_0 - C_xy_0)^2 = 0$$

If we make the substitutions

$$m^2 = (C_x - y_0)^2 + (C_y + x_0)^2$$

$$\eta^2 = (C_y - x_0)^2 + (C_x + y_0)^2 \qquad (m > 0, \quad \eta > 0)$$

Then the equation serving for the determination of the semi-axes of the ellipse takes the very simple form

$$S^2 - \tfrac{1}{2}(n^2 + m^2)S + \tfrac{1}{16}(n^2 - m^2)^2 = 0$$

The semi-axes of the ellipse are calculated very simply:

$$\bar{a} = \frac{n + m}{2} \qquad \bar{b} = \frac{|n - m|}{2}$$

and

$$\bar{e} = \frac{2\sqrt{nm}}{n + m} \qquad \cos\bar{\phi} = \frac{|n - m|}{n + m}$$

If the plane xy does not coincide with the plane of the orbit, then it is possible to obtain expressions for the semi-axes and by the well-known methods to determine the plane of the orbit, etc.

It is evident that the approximate formula obtained is most suitable, in the sense of smallness of error, in that part of the orbit where the heliocentric distance is close to the magnitude of the semimajor axis. This will be for values of the eccentric anomaly close to 90° or 270°.[3]

We shall clarify the connection of the exact solution with the simplest approximate solution. The coordinates of the planet can be expanded in series according to powers of time. This expansion is in a form very close to that which is obtained for the approximate solution. Therefore, we find the mean of our approximate solutions with expansions of the coordinates in series according to powers of time.

The expansion is identical for all three coordinates, and the approximate solutions are also of one type. Therefore, it is possible to limit oneself to a comparison of only one coordinate, for example, the abscissa. We have for the exact abscissa the following formula (2): $x = x_0F(\theta) + x_0'G(\theta)$, in which $F(\theta)$ and $G(\theta)$ are the functions determined, represented by infinite series according to powers of θ in which the coefficients contain the cube of the inverse distance and its derivative at the initial moment. Let $\theta = kt$, and, if t_0 is taken equal to 0, x_0' designates the derivative at the initial moment with respect to θ. We transform the written expression in order to make it approach the approximate solution. We shall replace k by $\mu \cdot a^{3/2}$, and then $\theta = \mu a^{3/2}t$. The derivative with respect to θ takes the form $x_0' = (x_0'/\mu)\cdot a^{-3/2}$, and the functions $F(\theta)$ and $G(\theta)$ are transformed to the form

$$F(\theta) = 1 - \frac{1}{2}\frac{a^3}{r_0^3}(\mu t)^2$$

$$G(\theta) = (\mu t)a^{3/2} - \frac{1}{6}\frac{a^{9/2}}{r_0^3}(\mu t)^3$$

If each series is limited to the first two terms with the same accuracy, the abscissa is expressed by

$$x = x_0\left[1 - \frac{1}{2}\frac{a^3}{r_0^3}(\mu t)^2\right] + \frac{\dot{x}_0}{\mu}\left[(\mu t) - \frac{1}{6}\frac{a^3}{r_0^3}(\mu t)^3\right]$$

We shall expand the cosine and sine in series by powers of the argument in our approximate formula and also limit ourselves to the first two terms. We obtain

$$\bar{x} = x_0\left[1 - \frac{1}{2}(\mu t)^2\right] + \frac{\dot{x}_0}{\mu}\left[(\mu t) - \frac{1}{6}(\mu t)^3\right]$$

From here the error of the approximate solution with the accepted accuracy can be written

$$x - \bar{x} = \frac{1}{2}x_0\left(1 - \frac{a^3}{r_0^3}\right)(\mu t)^2 + \frac{1}{6}\frac{\dot{x}_0}{\mu}\left(1 - \frac{a^3}{r_0^3}\right)(\mu t)^3$$

It is obvious from this equation that the errors will be small if the initial moment is chosen so that r_0 is approximately equal to a. The difference $1 - a^3r_0^{-3}$ can be estimated easily by the maximum and minimum values of r in elliptic motion:

$$\left|1 - \frac{a^3}{r_0^3}\right| \leqslant 3e + 6e^2 + \ldots$$

The modulus of the initial coordinates can be estimated above very simply, although rather crudely in many cases, by the quantity $a(1 + e)$. For the modulus of the components of velocity, divided by the mean motion, it is possible to write for the overstated estimate the form

$$\left|\frac{\dot{x}_0}{\mu}\right| \leqslant 2a\left(1 + e + \frac{3}{4}e^2\right)$$

It is necessary to note that this estimate will always be strongly overstated because it is accepted that the sine and cosine of the eccentric anomaly are transformed simultaneously as unity, as are the vectors **P** and **Q**, and the heliocentric distance r_0 has a minimum value. As a result it is possible to write the overstated estimate of the error of the approximate solution as

$$\left|\frac{x - \bar{x}}{a}\right| \leqslant (3e + 9e^2)\left[\frac{1}{2}(\mu t)^2 + \frac{1}{3}(\mu t)^3\right]$$

for orbits with small eccentricities. Simultaneously, the application of this estimate to all coordinates finally gives a very exaggerated estimate of the distance error.

The procedure for the calculations of the heliocentric coordinates is very simple and does not require much time. If the usual search ephemeris is calculated for moments of time close to the time of opposition, then, for the initial moment, it is necessary to take three or four of the usual moments being assumed. (This is essentially sufficient because, in the estimate of the error, it is always an important factor, depending on time.) Here, time t does not exceed 30. If the mean motion is close to 800″ per day, then the factor in the estimate of the error containing time does not exceed 0.0078. If the first of the ephemeris moments is taken for the initial moment, then this same factor reaches the magnitude 0.0227 for the sixth moment.

After the calculation of the direction cosines, it is necessary to determine the initial coordinates and their derivatives with respect to time, divided by the mean motion, by the usual formulas.

If a larger ephemeris is calculated and accuracy is needed in the fourth decimal, then for moderate eccentricity it is

[3] The attempted replacement of r by r_0 gave unsatisfactory results, because it is sufficiently accurate to obtain the coordinates over only 10 days. Although the distance changes slowly and smoothly, such changes in the differential equations can result in a noticeable change in the solution.

necessary to calculate the coordinates and components of velocity over 50–70 days by accurate formulas and to fill out scales of each moment in the ephemeris by the approximate formulas.

The calculation of the coordinates by the approximate formulas does not have to be explained.

We shall give examples of comparison ephemerides calculated according to the accurate and approximate formulas (Tables 1–4).

Table 1 Planet 807, Tseraskia (1953)

	June 12	June 22	July 2	July 12	July 22	August 1
x	0.8246	0.9134	1.001	1.088	1.174	1.259
$\bar{x}$	0.8246	0.9133	1.001	1.088	1.173	1.257
y	−2.949	−2.922	−2.892	−2.859	−2.825	−2.787
$\bar{y}$	−2.949	−2.921	−2.891	−2.857	−2.820	−2.781
z	−0.9578	−0.9636	−0.9684	−0.9725	−0.9756	−0.9779
$\bar{z}$	−0.9578	−0.9635	−0.9681	−0.9717	−0.9742	−0.9757

Table 2 Planet 1031, Arktika (1953)

	Nov. 29	Dec. 9	Dec. 19	Dec. 29	Jan. 8	Jan. 18
x	0.3689	0.2703	0.1714	0.07217	−0.02703	−0.1262
$\bar{x}$	0.3689	0.2703	0.1714	0.07227	−0.02688	−0.1260
y	2.907	2.915	2.920	2.921	2.919	2.914
$\bar{y}$	2.907	2.915	2.920	2.922	2.921	2.917
z	0.5694	0.5512	0.5323	0.5128	0.4928	0.4722
$\bar{z}$	0.5694	0.5512	0.5324	0.5130	0.4931	0.4727

In both examples, the first point is taken for the initial point in order to have less favorable conditions of accuracy. In spite of this, the errors seldom exceed several units in the fourth decimal, and only at separate points at the end of the ephemeris do the errors reach one or two units in the third decimal.

Table 3 Planet 881, Afina (1956)

	Nov. 3	Nov. 13	Nov. 23	Dec. 3	Dec. 13	Dec. 23
x	0.2477	0.1567	0.06541	−0.02592	−0.1172	−0.2083
$\bar{x}$	0.2476	0.1566	0.06541	−0.02592	−0.1172	−0.2083
y	2.530	2.563	2.592	2.618	2.641	2.661
$\bar{y}$	2.528	2.562	2.592	2.618	2.639	2.655
z	1.264	1.254	1.243	1.230	1.215	1.199
$\bar{z}$	1.263	1.254	1.243	1.229	1.214	1.196

Here, the third point is taken for the initial point, which decreases errors. The error reached six units in the fourth decimal in only one case. Thus, it is possible to say that at almost all points the approximate formulas give four decimals, with some uncertainty in the fourth decimal at remote points.

Table 4 Planet 1031, Arktika (1953–1954)

	Nov. 29	Dec. 9	Dec. 19	Dec. 29	Jan. 8	Jan. 18
x	0.3689	0.2703	0.1714	0.07217	−0.02703	−0.1262
$\bar{x}'$	0.3690	0.2703	0.1714	0.07222	−0.02697	−0.1281
y	2.907	2.915	2.920	2.921	2.919	2.914
$\bar{y}'$	2.907	2.915	2.920	2.921(1)	2.920	2.915
z	0.5694	0.5512	0.5323	0.5128	0.4928	0.4722
$\bar{z}'$	0.5695	0.5512	0.5323	0.5129	0.4929	0.4724

Here, for the initial point, we take the third ephemeris point because this limits the error. We limit ourselves to an accuracy of four decimals. A comparison with the second example shows that the choice of the initial point close to the middle of the ephemeris gives results noticeably better than in the case of the choice of the initial point at the beginning of the ephemeris.

The approximate solution considered has the same period as the exact solution.

3 First Refinement of the Simplest Approximate Solution

Although estimates of the error of the approximate solution would be noticeably overstated, nevertheless, it is quite possible to show that the error of this method has the order of the eccentricity multiplied by the square of the product of time by the mean motion. Since the mean motion must be expressed in radians per day in such an estimate, the product

μt is a small number until t becomes very great. If the eccentricity does not appear as a small number, and if the interval of time from the beginning up to the given moment is great, then the method turns out to be insufficiently accurate, even for a search ephemeris. For such cases, it is desirable to make the ephemeris more accurate, not passing on to accurate formulas. We shall also consider such a refinement. The method gives an inadequate approximation because we replaced r^{-3} by a^{-3} in the differential equations, even though a^{-3} also did not appear as the mean value of the quantity r^{-3}. Therefore, the simplest improvement of the approximate solution lies in the fact that the nonlinear factor r^{-3} in the differential equations is replaced by

$$b^{-3} = a^{-3}(1 - e^2)^{-3/2} = a^{-3}(1 + \tfrac{3}{2}e^2 + \tfrac{15}{8}e^4 + \ldots)$$

We obtain a system of differential equations, which again are solved independently of each other:

$$\frac{d^2\tilde{x}}{dt^2} + \nu^2\tilde{x} = 0 \qquad \frac{d^2\tilde{y}}{dt^2} + \nu^2\tilde{y} = 0$$

$$\frac{d^2\tilde{z}}{dt^2} + \nu^2\tilde{z} = 0 \qquad \nu^2 = \mu^2(1 - e^2)^{-3/2}$$

The second approximate solution which, on the average, will be considered to be more exact has the form

$$\tilde{x} = x_0 \cos\nu t + \frac{\dot{x}_0}{\nu} \sin\nu t$$

$$\tilde{y} = y_0 \cos\nu t + \frac{\dot{y}_0}{\nu} \sin\nu t$$

$$\tilde{z} = z_0 \cos\nu t + \frac{\dot{z}_0}{\nu} \sin\nu t$$

We assume also that the new approximate solution is an additional simpler approximate solution, which is applied when the initial distance (heliocentric) has a magnitude close to the magnitude of the semimajor axis of the ellipse. The second simpler solution must differ essentially from the first in that it has a period differing from the period of the exact solution. Therefore, such an approximation cannot be applied for the whole orbit, because, after the termination of the period, the approximate solution does not give the initial point of the trajectory. Even on the average, it must give a better approximation for the trajectory as a whole. We investigate at somewhat greater length the second simpler approximation. We denote

$$C_x' = \frac{\dot{x}_0}{\nu} \qquad C_y' = \frac{\dot{y}_0}{\nu} \qquad C_z' - \frac{\dot{z}_0}{\nu}$$

$$\frac{1}{\nu} = \frac{1}{\mu}(1 - e^2)^{3/4}$$

The elimination of time gives the equation of the plane of the orbit in the form

$$\begin{vmatrix} \tilde{x} & \tilde{y} & \tilde{z} \\ x_0 & y_0 & z_0 \\ C_x' & C_y' & C_z' \end{vmatrix} = 0$$

The plane of the orbit coincides with the plane of the exact orbit because the elements of the determinant C_x', C_y', C_z' differ from the elements C_x, C_y, C_z by the presence of the factor $(1 - e^2)^{3/4}$, generally for all three elements. Both approximate orbits are in contact at the initial point with the exact orbit. The dimensions of the second approximate ellipse are different from the dimensions of the first. It is seen from a comparison of ν^2 and μ^2 that this difference is the order of the square of the eccentricity. For an estimate of the error of the second approximate formula, we again make use of the expansion of the heliocentric coordinates in series according to powers of time. We limit ourselves to the first two terms, and we shall compare it with the expansion of the trigonometric functions in series, breaking off the latter after writing the first two terms. With the terms taken we obtain

$$x - \tilde{x} = \frac{1}{2} t^2 \left(\nu^2 - \mu^2 \frac{a^3}{r_0^3}\right)\left(x_0 + \frac{1}{3} t\dot{x}_0\right)$$

Analogous expressions can be written for the remaining coordinates. We shall expand the expression for ν^2 by powers of e and restrict ourselves to the first two terms. We make use of the inequality

$$a^3/r_0^3 \geqslant 1 - 3e + 6e^2$$

Following from the fact that the upper limit for r_0 is $a(1 + e)$, we obtain the following estimate of the error:

$$|x - \tilde{x}| \leqslant \frac{1}{2} t^2\mu^2 \left(3e - \frac{9}{\alpha} e^2\right) \left| x_0 + \frac{1}{3} \dot{x}_0 t \right|$$

which is suitable, finally, for smaller values of the eccentricity. An error of the order of the eccentricity multiplied by the square of $t\mu$ is obtained here also. The last quantity does not exceed 0.02 for the majority of small planets if t does not exceed 30 days. If the expression for μ in terms of a is taken into account, it is possible to show that the error is inversely proportional to the square of the semimajor axis of the Keplerian ellipse, because a is contained in the latter factor. The error of the second formula is smaller than the error of the first by a magnitude of the order of the square of the eccentricity, whereas both errors are the order of the first power of the eccentricity.

4 Second Refinement of the First Approximate Value

The first approximate value is obtained with an error of the order of the first power of the eccentricity, which also was shown in the estimate of the error. As indicated in the preceding paragraph, the refinement of the mean value does not give an actual improvement because, in the expansion of the partial average quantity, terms of the order of the eccentricity are rejected.

The natural refinement of the first approximation can be considered as the refinement that would be obtained if, in the replacement of the cube of the inverse distance, we take into account also such terms containing the first power of the eccentricity, that is, if we make use of the approximate equality (2):

$$\frac{1}{r^3} = \frac{1}{a^3}(1 + 3e \cos M)$$

It is possible to show that the approximation obtained will contain an error of the order of the square of the eccentricity. After the indicated replacements, we obtain the approximate equation

$$\frac{d^2x}{dt^2} + \frac{k^2x}{a^3}(1 + 3e \cos M) = 0$$

and two others for the coordinates y and z which have exactly the same form. The problem is simplified because a decoupled system of differential equations of the second order is obtained, but this simplification is inadequate because the equations contain variable coefficients. We shall apply the simplification of the system. We copy the differential equation in the form

$$d^2x/dt^2 + \mu^2x = -3e\mu^2x \cos M$$

and on the right sides we replace the coordinates by their expressions from the first approximation. Since the right sides contain a factor of the eccentricity, and the values of the

Table 5 Planet 1031, Arktika (1953–1954)

	Nov. 29	Dec. 9	Dec. 19	Dec. 29	Jan. 8	Jan. 18
x	0.36890	0.27029	0.17135	0.07217	−0.02703	−0.12618
$\bar{x}$	0.36895	0.27030	0.17135	0.07222	−0.02696	−0.12614
$\tilde{x}$	0.36891	0.27029	0.17135	0.07221	−0.02698	−0.12617
y	2.9069	2.9149	2.9197	2.9211	2.9193	2.9141
$\bar{y}$	2.9073	2.9151	2.9197	2.9213	2.9198	2.9153
$\tilde{y}$	2.9068	2.9150	2.9197	2.9212	2.9193	2.9141
z	0.56938	0.55115	0.53230	0.51283	0.49279	0.47220
$\bar{z}$	0.56947	0.55118	0.53230	0.51287	0.49290	0.47241
$\tilde{z}$	0.56938	0.55116	0.53230	0.51285	0.49281	0.47220

coordinates in the result of the averaging are obtained with an error of the order of the eccentricity, then the error from the proposed transformations will have the order of the square of the eccentricity, that is, the same order as the error due to replacement of the cube of the inverse distance. After this we obtain a decoupled system of linear nonhomogeneous differential equations having the form

$$\frac{d^2x}{dt^2} + \mu^2 x = -3e\mu^2 \left(x_s \cos\mu\tau + \frac{\dot{x}_s}{\mu} \sin \mu\tau\right) \cos M$$

$$\tau = t - t_s$$

where t_s is the initial moment.

The equations for the other two coordinates are constructed exactly the same way. If M_s designates the mean anomaly at the initial moment, then we have $M = M_s + \mu\tau$. To take this expression into account, we transcribe the right part of the differential equation into the form

$$-\tfrac{3}{2}e\mu^2[(x_s \cos M_s - C_x \sin M_s) + (x_s \cos M_s + C_x \sin M_s) \cos 2\mu\tau + (C_x \cos M_s - x_s \sin M_s) \sin 2\mu\tau]$$

with analogous expressions for the other two equations. The quantities $\dot{x}_s/\mu$, etc., are designated by C_x, C_y, C_z.

We obtain the solution of the differential equations by known methods, satisfying the initial conditions, in the form

$$x = \bar{x} - \tfrac{3}{2}e(x_s \cos M_s - C_x \sin M_s) + e(x_s \cos M_s + C_x \sin M_s) \cos\mu\tau + e(x_s \sin M_s - C_x \cos M_s) \sin\mu\tau + \tfrac{1}{2}e(x_s \cos M_s + C_x \sin M_s) \cos 2\mu\tau + \tfrac{1}{2}e(C_x \cos M_s - x_s \sin M_s) \sin 2\mu\tau$$

Analogous expressions are found for y and z. In these expressions $\bar{y}$, $\bar{z}$ denote the solutions of the first approximation, that is, those which are obtained in the foregoing as a result of the partial averages in the differential equations.

We introduce the notations

$$x_s \cos M_s - C_x \sin M_s = 2\alpha x$$

$$x_s \cos M_s + C_x \sin M_s = 2\beta_x$$

$$x_s \sin M_s - C_x \cos M_s = 2\gamma_x$$

and analogously for α_y, β_y, γ_y, α_z, β_z, γ_z. The calculation of the quantities α, β, γ with the values of x, y, z is easily controlled, for example, according to the formulas

$$4\beta_x^2 + 4\gamma_x^2 = x_s^2 + C_x^2$$

$$\alpha_x(C_x^2 + x_s^2) + \beta_x(C_x^2 - x_s^2) + \gamma_x \cdot 2C_x x_s = 0$$

The first is evident. The second is obtained by the elimination of $\cos M_s$ and $\sin M_s$ from the equalities determining α, β, γ.

After the calculations of the quantities α, β, γ, the solution of the equations can be written in the form

$$\tilde{x} = \bar{x} + eR_x(\tau) \qquad \tilde{y} = \bar{y} + eR_y(\tau) \qquad \tilde{z} = \bar{z} + eR_z(\tau)$$

where $R_x(\tau)$, $R_y(\tau)$, and $R_z(\tau)$ are determined by the formulas

$$R_x(\tau) = -3\alpha_x + (3\alpha_x - \beta_x) \cos\mu\tau + 2\gamma_x \sin\mu\tau + \beta_x \cos 2\mu\tau - \gamma_x \sin 2\mu\tau$$

where R_y and R_z are determined by analogous formulas.

We consider an example of the application of the written formulas to the calculation of an ephemeris for the planet Arktika (Table 5). We reduce the values of the coordinates calculated according to the exact formulas, the approximate values $\bar{x}$, $\bar{y}$, $\bar{z}$, and the refined values with errors of the order of the square of the eccentricity. The initial point is close to the middle (the third).

Here x is exact (calculated approximately but according to exact formulas), and $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ are refined with errors of the order of the square of the eccentricity.

5 Order of Calculations and the Summary of Formulas

If the usual ephemeris is required for six moments, then the rectangular heliocentric coordinates and their derivatives are calculated, according to known formulas, divided by the mean motion according to formulas of the form

$$C_x = \frac{\dot{x}_3}{\mu} = \frac{a}{r_3} (-aP_x \sin E_3 + bQ_x \cos E_3)$$

For C_y and C_z there are analogous formulas. P and Q are the projection coefficients. In making use of the approximate formulas, it is better to consider the ephemeris for five moments.

If the ephemeris is calculated for a greater number of epochs (for example, in the calculation of perturbed motion by the method of Encke), then in the formulas mentioned it is necessary to calculate the coordinates and their derivatives for equidistant epochs with steps of 40 days. If the ephemeris is needed with the usual step of 10 days, then the coordinates are calculated according to the approximated formula

$$x(t) = x_s \cos\mu\tau + C_x \sin\mu\tau \qquad \tau = t - t_s$$

where t_s is the initial moment, in particular for $s = 3$. The formulas for the remaining coordinates are analogous. The refinement, if it is required, is made according to the formulas of Sec. 4.

—*Submitted April 11, 1958*

References

1 Subbotin, M. F., *A Course in Celestial Mechanics* (ONTI, United Scientific and Technical Press, Moscow-Leningrad, 1937), Vol. 2.

2 Subbotin, M. F., *A Course in Celestial Mechanics* (Gostechizdat, State Technical Press, Moscow-Leningrad, 1941), 2nd ed., Vol. 1.

Reviewer's Comment

Ephemerides for unperturbed elliptic motion are generally computed from the exact solution of the equations of motion via Kepler's equation. In his book on orbit determination (1), Dubyago describes a method of computing ephemerides based directly on numerical integration of the unperturbed two-body equations of motion. Shchigolev's paper discusses a computational method for obtaining an approximate solution to the two-body equations of motion which is much simpler than the other two methods.

Shchigolev replaces the factor $1/r^3$ in the equations of motion by its expected value. He proposes several different methods for determining this expected value.

This substitution linearizes and decouples the equations of motion, permitting an analytic solution to be obtained. Several refinements of this technique are discussed and applied to minor planet orbits.

The methods used by Shchigolev would probably be most useful for computing search ephemerides for minor planets and comets where great precision is not required. They permit the rapid calculation of an ephemeris by a desk calculator when a digital computer is not available. His methods have also been applied to space vehicle orbit transfer computations (2,3).

—Frederick T. Smith
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1 Dubyago, A. D., *The Determination of Orbits* (The Macmillan Company, New York, 1961); English translation.
2 Smith, F. T., "A discussion of a midcourse guidance technique for space vehicles," RAND Corp. Research Memo. RM-2581 (October 3, 1960).
3 Hutcheson, J. H. and Smith, F. T., "An orbital control process for a 24-hour communication satellite," RAND Corp. Research Memo. RM-2809-NASA (October 1961).

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Digest of Translated Russian Literature

The following abstracts have been selected by the Editor from translated Russian journals supplied by the indicated societies and organizations, whose cooperation is gratefully acknowledged. Information concerning subscriptions to the publications may be obtained from these societies and organizations. Note: Volumes and numbers given are those of the English translations, not of the original Russian.

JOURNAL OF PHYSICAL CHEMISTRY (Zhurnal Fizicheskoi Khimii). Published by The Chemical Society, London.

Volume 35, number 10, October 1961.

Thermodynamic Functions of Monatomic and Diatomic Gases Over a Wide Temperature Range. III. Nitrogen Atoms, Nitrogen Molecules, and Nitric Oxide in the Ideal State up to 20,000°K, V. S. Yungman, L. V. Gurvich, V. A. Kvlividze, E. A. Prozorovskii, and N. P. Rtishcheva, pp. 1073–1077.

The present paper describes the calculation of the thermodynamic functions (Φ_T^*, S_T^0, $H_T^0 - H_0^0$) of N, N_2, and NO in the ideal state at 1 atm pressure between 293.15° and 20,000°K, by methods described previously. Equilibrium constants K_p for dissociation into monatomic gases have also been calculated for N_2 and NO.

Summary:

1 A selection has been made of the most reliable values of the molecular constants of N, N_2, and NO necessary for the accurate calculation of the thermodynamic functions of these gases.

2 The thermodynamic functions of the gases N, N_2, and NO in the ideal state at 1 atm pressure between 293.15° and 20,000°K have been calculated by direct summation over the energy levels on the BESM of the USSR Academy of Sciences.

3 The dissociation constants of N_2 and NO between 293.15° and 20,000°K have been calculated.

Isotope-Exchange Method for Measuring Vapor Pressures and Diffusion Coefficients. III. Treatment of Experimental Data, V. I. Lozgachev, pp. 1084–1090.

Summary: A procedure has been developed for the determination of constants in the solution of the diffusion equation for isotope exchange through the gaseous phase in γ- and β-radiation measurements. Equations have been found which make it possible to find from one experimental curve, plotted from the start of the process up to the steady state, the rate of evaporation n_0, the diffusion coefficient D, and the thickness of the exchange layer δ. A computation formula is proposed as well as a method for the experimental determination of the condensation coefficient α.

Theory of Electrical Transport. II. Multicomponent Metallic Systems, D. K. Belashchenko and B. S. Bokshtein, pp. 1099–1101.

The previous paper in this series applied the thermodynamics of irreversible processes to the diffusion of the components of a binary metallic alloy when a direct current is passed through it. It is useful to generalize these results to multicomponent metallic systems. The present paper concerns a three-component system, assuming for simplicity that the partial volumes of the components are equal.

Oxidation-Reduction Kinetics of Hydrogen, Oxygen, and a Stoichiometric Oxygen-Hydrogen Mixture at a Platinum Electrode in Electrolyte Solutions, K. I. Rozental' and V. I. Veselovskii, pp. 1114–1118.

It has been shown in our laboratories that the oxidizing and reducing components (hydroxyl radicals, hydrogen atoms, O_2, H_2, and H_2O_2 molecules), resulting from the radiolysis of water, produce characteristic electrochemical processes at the electrode. When this occurs, the electrode potential may acquire any value between the potentials of the hydrogen and oxygen electrodes, depending on the properties of the electrode metal, its reaction with the radiolysis products, its adsorption capacity with respect to them, and the rate of ionization of the given substance at the electrode. Therefore, an investigation of the kinetics of the electrochemical interaction between oxygen and hydrogen on metal electrodes in electrolyte solutions acquires an additional interest.

Earlier investigations show that the catalysis of the oxidation of oxygen-hydrogen mixtures in electrolyte solutions is determined by the electrochemical properties of the catalyst and the electrode potential.

In our studies (by anodic polarography) of the oxidation and reduction of gaseous H_2-O_2 mixtures on a platinum electrode in electrolyte solutions, it has been found that the effectiveness of the process depends, to a large extent, upon the potential of the electrode, nature of the anion, and the pH of the solution.

The method employed here makes it possible to measure directly, over a wide range of potentials applied by polarization of the Pt electrode (0–1.6 v), the true rates of the oxidation of